A CHARACTERIZATION OF THE QUANTUM COHOMOLOGY RING OF G/B AND APPLICATIONS

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Abstract. We show that the small quantum product of the generalized flag manifold G/B is a product operation on $H^*(G/B)\otimes\mathbb{R}[q_1,\ldots,q_l]$ uniquely determined by the fact that it is a deformation of the cup product on $H^*(G/B)$, it is commutative, associative, graded with respect to $\deg(q_i)=4$, it satisfies a certain relation (of degree two), and the corresponding Dubrovin connection is flat. We deduce that it is again the flatness of the Dubrovin connection which characterizes essentially the solutions of the "quantum Giambelli problem" for G/B. This result gives new proofs of the quantum Chevalley formula (see D. Peterson [Pe] and Fulton and Woodward [Fu-Wo]), and of Fomin, Gelfand and Postnikov's description of the quantization map for Fl_n (see [Fo-Ge-Po]).

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1. Introduction

Let us consider the complex flag manifold G/B, where G is a connected, simply connected, simple, complex Lie group and $B\subset G$ a Borel subgroup. Let $\mathfrak{t}$ be the Lie algebra of a maximal torus of a compact real form of G and $\Phi\subset\mathfrak{t}^*$ the corresponding set of roots. Consider an arbitrary W-invariant inner product $\langle\ ,\ \rangle$ on $\mathfrak{t}$. To any root α corresponds the coroot

$$\alpha^\vee := \frac{2\alpha}{\langle\alpha,\alpha\rangle}$$

which is an element of $\mathfrak{t}$, by using the identification of $\mathfrak{t}$ and $\mathfrak{t}^*$ induced by $\langle\ ,\ \rangle$. If $\{\alpha_1,\ldots,\alpha_l\}$ is a system of simple roots then $\{\alpha_1^\vee,\ldots,\alpha_l^\vee\}$ is a system of simple coroots. Consider $\{\lambda_1,\ldots,\lambda_l\}\subset\mathfrak{t}^*$ the corresponding system of fundamental weights, which are defined by $\lambda_i(\alpha_j^\vee)=\delta_{ij}$. The Weyl group W is the subgroup of $O(\mathfrak{t},\langle\ ,\ \rangle)$ generated by the reflections about the hyperplanes $\ker\alpha$, $\alpha\in\Phi^+$. It can be shown that W is in fact generated by the *simple reflections* $s_1=s_{\alpha_1},\ldots,s_l=s_{\alpha_l}$ about the hyperplanes $\ker\alpha_1,\ldots,\ker\alpha_l$. The *length* $l(w)$ of w is the minimal number of factors in a decomposition of w as a product of simple reflections. We denote by w_0 the longest element of W.

Let $B^-\subset G$ denote the Borel subgroup opposite to B. To each $w\in W$ we assign the *Schubert variety* $X_w=\overline{B^-.w}$. The Poincaré dual of $[X_w]$ is an element of $H^{2l(w)}(G/B)$, which is called the *Schubert class*. The set $\{\sigma_w \mid w\in W\}$ is a basis of $H^*(G/B)=H^*(G/B,\mathbb{R})$, hence $\{\sigma_{s_1},\ldots,\sigma_{s_l}\}$ is a basis of $H^2(G/B)$. A theorem of Borel [Bo] says that the map

$$H^*(G/B)\to S(\mathfrak{t}^*)/S(\mathfrak{t}^*)^W=\mathbb{R}[\{\lambda_i\}]/I_W \tag{1}$$

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described by $\sigma_{s_i} \mapsto [\lambda_i]$, $1 \le i \le l$, is a ring isomorphism (we are denoting by $S(\mathfrak{t}^*)^W = I_W$ the ideal of $S(\mathfrak{t}^*) = \mathbb{R}[\{\lambda_i\}]$ generated by the non-constant W-invariant polynomials). We will frequently identify $H^*(G/B)$ with the quotient ring from above.

To any l-tuple $d = (d_1, \dots, d_l)$ with $d_i \in \mathbb{Z}$, $d_i \ge 0$ corresponds a *Gromov-Witten invariant*. This assigns to any three Schubert classes $\sigma_u, \sigma_v, \sigma_w$ the number denoted by $\langle \sigma_u | \sigma_v | \sigma_w \rangle_d$, which counts the holomorphic curves $\varphi : \mathbb{C}P^1 \to G/B$ such that $\varphi_*([\mathbb{C}P^1]) = d$ in $H_2(G/B)$ and $\varphi(0)$, $\varphi(1)$ and $\varphi(\infty)$ are in general translates of the Schubert varieties dual to σ_u, σ_v, respectively σ_w. Let us consider the variables $q_1, \dots, q_l$. The *quantum cohomology ring* of G/B is the space $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ equipped with the product $\circ$ which is $\mathbb{R}[\{q_i\}]$-linear and for any two Schubert classes σ_u, σ_v, $u, v \in W$ we have

$$\sigma_u \circ \sigma_v = \sum_{d=(d_1,\dots,d_l)\ge 0} q^d \sum_{w \in W} (\sigma_u \circ \sigma_v)_d \sigma_w,$$

$u, v \in W$. Here q^d denotes $q_1^{d_1} \dots q_l^{d_l}$ and the cohomology class $(\sigma_u \circ \sigma_v)_d$ is determined by

$$\langle (\sigma_u \circ \sigma_v)_d, \sigma_w \rangle = \langle \sigma_u | \sigma_v | \sigma_w \rangle_d, \tag{2}$$

for any $w \in W$. It turns out that the product $\circ$ is commutative, associative and it is a deformation of the cup product (by which mean that if we formally set $q_1 = \dots = q_l = 0$, then $\circ$ becomes the same as the cup product). If we assign

$$\deg q_i = 4, \quad 1 \le i \le l,$$

then we also have the grading condition

$$\deg(a \circ b) = \deg a + \deg b,$$

for any two homogeneous elements a, b of $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$. For more details about quantum cohomology we refer the reader to Fulton and Pandharipande [Fu-Pa].

The *Dubrovin connection* attached to the quantum product defined above is a connection[1] $\nabla^\hbar$ on the trivial vector bundle $H^*(G/B) \times H^2(G/B) \to H^2(G/B)$ defined as follows: Denote by $t_1, \dots, t_l$ the coordinates on $H^2(G/B)$ induced by the basis $\sigma_{s_1}, \dots, \sigma_{s_l}$. Consider the 1-form ω on $H^2(G/B)$ with values in $\mathrm{End}(H^*(G/B))$ given by

$$\omega_t(X, Y) = X \circ Y,$$

for $t = (t_1, \dots, t_l) \in H^2(G/B)$, $X \in H^2(G/B)$ and $Y \in H^*(G/B)$, where the convention

$$q_i = e^{t_i}, \quad 1 \le i \le l$$

is in force. Finally set

$$\nabla^\hbar = d + \frac{1}{\hbar}\omega.$$

Note that the 1-form ω can be expressed as

$$\omega = \sum_{i=1}^{l} \omega_i dt_i,$$

[1]More precisely, a family of connections depending on the parameter $\hbar \in \mathbb{R} \setminus \{0\}$.

where ω_i denotes the matrix of the operator $\sigma_{s_i}\circ$ on $H^*(G/B)$ with respect to the basis consisting of the Schubert classes. The following result is well-known (cf. [Du]):

Lemma 1.1. *The Dubrovin connection $\nabla^\hbar$ is flat for any $\hbar \in \mathbb{R}\setminus\{0\}$, i.e. we have*

$$d\omega = \omega \wedge \omega = 0 \tag{3}$$

Proof. The fact that $d\omega = 0$ follows from

$$\frac{\partial}{\partial t_i}\omega_j = \frac{\partial}{\partial t_j}\omega_i,$$

which is equivalent to

$$d_i(\sigma_{s_j}\circ\sigma_w)_d = d_j(\sigma_{s_i}\circ\sigma_w)_d$$

for any $w \in W$ and any $d = (d_1, \ldots, d_l)$, hence, by (2), to

$$d_i\langle\sigma_{s_j}|\sigma_w|\sigma_v\rangle_d = d_j\langle\sigma_{s_i}|\sigma_w|\sigma_v\rangle_d.$$

The latter equality follows from the "divisor property" (see [Fu-Pa, equation (40)] for a more general version of this formula):

$$\langle\sigma_{s_j}|\sigma_w|\sigma_v\rangle_d = d_j\langle\sigma_w|\sigma_v\rangle_d.$$

The equality $\omega\wedge\omega = 0$, i.e. $\omega_i\omega_j = \omega_j\omega_i$, $1 \le i, j \le l$, follows from the fact that the product $\circ$ is commutative and associative. □

Another important property of the quantum product which is of interest for us is that we have the relation:

$$\sum_{i,j=1}^{l}\langle\alpha_i^\vee, \alpha_j^\vee\rangle\sigma_{s_i}\circ\sigma_{s_j} = \sum_{i=1}^{l}\langle\alpha_i^\vee, \alpha_i^\vee\rangle q_i. \tag{4}$$

In order to prove this we take into account that:

- we have (see [Kim] or [Ma1, Lemma 3.2])
$$\sigma_{s_i}\circ\sigma_{s_j} = \sigma_{s_i}\sigma_{s_j} + \delta_{ij}q_j$$
- the polynomial $\sum_{i,j=1}^{l}\langle\alpha_i^\vee, \alpha_j^\vee\rangle\lambda_i\lambda_j \in S(\mathfrak{t}^*)$ is W-invariant (being just the squared norm on $\mathfrak{t}$); hence, according to (1), the following relation holds in $H^*(G/B)$:
$$\sum_{i,j=1}^{l}\langle\alpha_i^\vee, \alpha_j^\vee\rangle\sigma_{s_i}\sigma_{s_j} = 0.$$

The goal of this paper is to show that the quantum product for G/B is essentially determined by the equations (3) (the flatness of the Dubrovin connection) and (4) (the degree two relation). More precisely, we will prove that:

Theorem 1.2. *Let $\star$ be a product on the space $H^*(G/B)\otimes\mathbb{R}[\{q_i\}]$ which is commutative, associative, is a deformation of the cup product (in the sense defined above), satisfies the condition $\deg(a\star b) = \deg a + \deg b$, for a, b homogeneous elements of $H^*(G/B)\otimes\mathbb{R}[\{q_i\}]$, with respect to the grading $\deg q_i = 4$, and*

(a) *the Dubrovin connection* $\nabla^\hbar = d + \frac{1}{\hbar}\omega$, *with* $\omega(X,Y) = X \star Y$ *is flat. In other words, if* ω_k *is the matrix of the* $\mathbb{R}[\{q_i\}]$*-linear endomorphism* $\sigma_{s_k}\star$ *of* $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ *with respect to the Schubert basis, then we have*

$$\frac{\partial}{\partial t_i}\omega_j = \frac{\partial}{\partial t_j}\omega_i$$

for all $1 \le i,j \le l$ *(the convention* $q_i = e^{t_i}$ *is in force).*

(b) *we have*

$$\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle \sigma_{s_i} \star \sigma_{s_j} = \sum_{i=1}^{l} \langle \alpha_i^\vee, \alpha_i^\vee \rangle q_i.$$

Then $\star$ *is the quantum product* $\circ$.

The proof will be done in section 2. There are two corollaries we would like to deduce from this theorem. The first one is a characterization of the quantum Giambelli polynomials in terms of the flatness of the Dubrovin connection. More precisely, let us denote by $QH^*(G/B)$ the quotient ring $\mathbb{R}[\{\lambda_i\},\{q_i\}]/\langle R_1,\ldots,R_l\rangle$, where $R_1,\ldots,R_l$ are the quantum deformations in the quantum cohomology ring $(H^*(G/B)\otimes\mathbb{R}[\{q_i\}],\circ)$ of the fundamental homogeneous generators of $S(\mathfrak{t}^*)^W$ ($R_1,\ldots,R_l$ have been determined explicitly by B. Kim in [Kim]; we will present in section 2 a few more details about that). For any $c \in \mathbb{R}[\{\lambda_i\},\{q_i\}]$ we denote by $[c]_q$ the coset of c in $QH^*(G/B)$. The map $\sigma_{s_i} \mapsto [\lambda_i]_q$ induces a tautological isomorphism

$$(H^*(G/B)\otimes\mathbb{R}[\{q_i\}],\circ) \simeq QH^*(G/B). \tag{5}$$

Finding for each $w \in W$ a polynomial $\hat{c}_w \in \mathbb{R}[\{\lambda_i\},\{q_i\}]$ whose coset in $QH^*(G/B)$ is the image of σ_w — in other words, solving the quantum Giambelli problem — would lead to a complete knowledge of the quantum cohomology of G/B. We are looking for conditions which determine the polynomials $\hat{c}_w$. First of all, let us consider for each $w \in W$ a polynomial[2] $c_w \in \mathbb{R}[\{\lambda_i\}]$ whose coset corresponds to σ_w via the isomorphism (1). There are two natural conditions that we impose to the polynomials $\hat{c}_w$:

$$\deg \hat{c}_w = \deg c_w \tag{6}$$

with respect to the grading $\deg \lambda_i = 2$, $\deg q_i = 4$, and

$$\hat{c}_w|_{(\text{all } q_i = 0)} = c_w. \tag{7}$$

Whenever the conditions (6) and (7) are satisified, the cosets $[\hat{c}_w]_q$, $w \in W$, are a basis of $QH^*(G/B)$ over $\mathbb{R}[\{q_i\}]$. Consider the 1-form

$$\omega = \sum_{i=1}^{l} \omega_i dt_i,$$

where ω_i is the matrix of multiplication of $QH^*(G/B)$ by $[\lambda_i]_q$ with respect to the latter basis. We can prove that:

[2] These are solutions of the classical Giambelli problem for G/B. Such polynomials have been constructed for instance by Bernstein, I. M. Gelfand and S. I. Gelfand in [Be-Ge-Ge].

Corollary 1.3. *Let $\hat{c}_w$, $w \in W$, be polynomials in $\mathbb{R}[\{\lambda_i\},\{q_i\}]$ which satisfy the properties (6) and (7). Then the image of σ_w by the isomorphism (5) is $[\hat{c}_w]_q$ for all $w \in W$ if and only if the connection*

$$\nabla^{\hbar} = d + \frac{1}{\hbar}\omega$$

is flat for all $\hbar \in \mathbb{R} \setminus \{0\}$. The latter condition reads

$$\frac{\partial}{\partial t_i}\omega_j = \frac{\partial}{\partial t_j}\omega_i,$$

for all $1 \le i, j \le l$.

Proof. Consider the $\mathbb{R}[\{q_i\}]$-linear isomorphism[3]

$$\delta : QH^*(G/B) \to H^*(G/B) \otimes \mathbb{R}[\{q_i\}] = \mathbb{R}[\{\lambda_i\},\{q_i\}]/(I_W \otimes \mathbb{R}[\{q_i\}])$$

determined by

$$\delta[\hat{c}_w]_q = [c_w], \tag{8}$$

for all $w \in W$. Define the product $\star$ on $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ by

$$x \star y = \delta(\delta^{-1}(x)\delta^{-1}(y)),$$

$x, y \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$. The product is commutative, associative, it is a deformation of the cup product on $H^*(G/B)$, and it satisfies $\deg(a \star b) = \deg a + \deg b$, where $a, b \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ are homogeneous elements. The map δ is obviously a ring isomorphism between $QH^*(G/B)$ and $(H^*(G/B) \otimes \mathbb{R}[\{q_i\}], \star)$. In particular, the following degree two relation holds:

$$\sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle [\lambda_i] \star [\lambda_j] = \sum_{i=1}^{l} \langle \alpha_i^\vee, \alpha_i^\vee \rangle q_i.$$

Moreover, the matrix of $[\lambda_i]\star$ on $H^*(G/B) \otimes \mathbb{R}[q_1, \ldots, q_l]$ with respect to the Schubert basis $\{[c_w] : w \in W\}$ is just ω_i. So if the connection $\nabla^{\hbar}$ is flat for all $\hbar$, then, by Theorem 1.2, the products $\star$ and $\circ$ are the same. This implies that δ is just the isomorphism (5). The conclusion follows from the definition (8) of δ. $\square$

Corollary 1.3 will be used in section 3 in order to recover the "quantization via standard monomials" theorem of Fomin, Gelfand, and Postnikov (see [Fo-Ge-Po, Theorem 1.1]).

Our second application of Theorem 1.2 concerns the combinatorial quantum product $\star$ on $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$, which has been constructed in [Ma4]. By definition, this product satisfies the following *quantum Chevalley formula*:

$$\sigma_{s_i} \star \sigma_w = \sigma_{s_i}\sigma_w + \sum \lambda_i(\alpha^\vee)\sigma_{ws_\alpha},$$

[3] This is what Amarzaya and Guest [Am-Gu] call a "quantum evaluation map".

for $1 \le i \le l$, $w \in W$. Here the sum runs over all positive roots α with the property that $l(ws_\alpha) = l(w) - 2\mathrm{height}(\alpha^\vee) + 1$, where we consider the expansion $\alpha^\vee = m_1\alpha_1^\vee + \ldots + m_l\alpha_l^\vee$, $m_j \in \mathbb{Z}$, $m_j \ge 0$ and denote

$$\mathrm{height}(\alpha^\vee) = m_1 + \ldots + m_l, \quad q^{\alpha^\vee} = q_1^{m_1} \ldots q_l^{m_l}.$$

We have also showed in [Ma4] that $\star$ satisfies all hypotheses of Theorem 1.2. We deduce:

Corollary 1.4. *The combinatorial and actual quantum products coincide. Consequently, the quantum product $\circ$ satisfies the quantum Chevalley formula:*

$$\sigma_{s_i} \circ \sigma_w = \sigma_{s_i}\sigma_w + \sum_{l(ws_\alpha)=l(w)-2\mathrm{height}(\alpha^\vee)+1} \lambda_i(\alpha^\vee)\sigma_{ws_\alpha}, \tag{9}$$

for $1 \le i \le l$, $w \in W$.

Remarks. 1. The formula (9) plays a crucial role in the study of the quantum cohomology algebra of G/B, as this is generated over $\mathbb{R}[q_1, \ldots, q_l]$ by the degree 2 Schubert classes $\sigma_{s_1}, \ldots, \sigma_{s_l}$. The formula was announced by D. Peterson in [Pe]. A rigorous intersection-theoretic proof has been given by W. Fulton and C. Woodward in [Fu-Wo]. It is one of the aims of our paper to give an alternative, conceptually new, proof of this formula.

2. The tool we will be using in the proof of Theorem 1.2 is the notion of $\mathcal{D}$-module, in the spirit of Guest [Gu], Amarzaya and Guest [Am-Gu], and Iritani [Ir]. More precisely, we will show that the $\mathcal{D}$-modules associated in Iritani's manner to the products $\circ$ and $\star$ are isomorphic, and then we conclude by using a certain uniqueness argument of Amarzaya and Guest [Am-Gu] (for more details, see the next section).

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2. $\mathcal{D}$-modules and quantum cohomology

The goal of this section is to give a proof of Theorem 1.2.

We denote by $\mathcal{D}$ the Heisenberg algebra, by which we mean the associative $\mathbb{R}[\hbar]$-algebra generated by $Q_1, \ldots, Q_l$, $P_1, \ldots, P_l$, subject to the relations

$$[Q_i, Q_j] = [P_i, P_j] = 0, \quad [P_i, Q_j] = \delta_{ij}\hbar Q_j, \tag{10}$$

$1 \le i, j \le l$. It becomes a graded algebra with respect to the assignments

$$\deg Q_i = 4, \quad \deg P_i = \deg \hbar = 2. \tag{11}$$

Note that any element D of $\mathcal{D}$ can be written uniquely as an $\mathbb{R}[\hbar]$-linear combination of monomials of type $Q^I P^J$.

A concrete realization of $\mathcal{D}$ can be obtained by putting $Q_i = e^{t_i}$ and $P_i = \hbar\frac{\partial}{\partial t_i}$, $1 \le i \le l$. We will be interested in certain elements of $\mathcal{D}$ which arise in connection with the Hamiltonian system of Toda lattice type corresponding to the coroots of G, namely the first quantum

integrals of motion of this system. Those are homogeneous elements $D_k = D_k(\{Q_i\}, \{P_i\}, \hbar)$ of $\mathcal{D}$, $1 \le k \le l$, which commute with

$$D_1 = \sum_{i,j=1}^{l} \langle \alpha_i^\vee, \alpha_j^\vee \rangle P_i P_j - \sum_{i=1}^{l} \langle \alpha_i^\vee, \alpha_i^\vee \rangle Q_i$$

and also satisfy the property that $D_k(\{0\}, \{\lambda_i\}, 0)$, $1 \le k \le l$, are just the fundamental homogeneous W-invariant polynomials (for more details concerning the differential operators $D_1, \dots, D_l$ we address the reader to [Ma3]). We will denote by $\mathcal{I}$ the left sided ideal of $\mathcal{D}$ generated by $D_1, \dots, D_l$.

Let $\star$ be a product on $H^*(G/B) \otimes \mathbb{R}[\{q_i\}]$ which satisfies the hypotheses of Theorem 1.2. Let us denote by E the $\mathcal{D}$-module (i.e. vector space with an action of the algebra $\mathcal{D}$) $H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ defined by

$$Q_i.a = q_i a, \quad P_i.a = \sigma_{s_i} \star a + \hbar q_i \frac{\partial}{\partial q_i} a,$$

$1 \le i \le l$, $a \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$. The isomorphism type of the $\mathcal{D}$-module E corresponding to $\star$ is uniquely determined by the hypotheses of Theorem 1.2, as the following proposition shows:

Proposition 2.1. *If $\star$ is a product with the properties stated in Theorem 1.2, then the map $\phi : \mathcal{D} \to H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ given by*

$$f(\{Q_i\}, \{P_i\}, \hbar) \overset{\phi}{\mapsto} f(\{Q_i\}, \{P_i\}, \hbar).1 = f(\{q_i\}, \{\sigma_{s_i} \star + \hbar q_i \frac{\partial}{\partial q_i}\}, \hbar).1$$

is surjective and induces an isomorphism of $\mathcal{D}$-modules

$$\mathcal{D}/\mathcal{I} \simeq E, \tag{12}$$

where $\mathcal{I}$ is the left sided ideal of $\mathcal{D}$ generated by the quantum integrals of motion of the Toda lattice (see above).

Proof. We will use the grading on $H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ induced by the usual grading on $H^*(G/B)$, $\deg q_i = 4$ and $\deg \hbar = 2$. Combined with the grading defined by (11), this makes ϕ into a degree preserving map (more precisely, it maps a homogeneous element of $\mathcal{D}$ to a homogeneous element of the same degree in $H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$).

Let us prove first the surjectivity stated in our theorem. It is sufficient to show that any homogeneous element $a \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ can be written as $f(\{Q_i\}, \{P_i\}, \hbar).1$. We proceed by induction on $\deg a$. If $\deg a = 0$, everything is clear. Now consider $a \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ a homogeneous element of degree at least 2. By a result of Siebert and Tian [Si-Ti], we can express

$$a = g(\{q_i\}, \{\sigma_{s_i} \star\}, \hbar)$$

for a certain polynomial g. We have

$$a - g(\{Q_i\}, \{P_i\}, \hbar).1 = a - g(\{q_i\}, \{\sigma_{s_i} \star + \hbar q_i \frac{\partial}{\partial q_i}\}, \hbar).1 = \hbar b,$$

where $b \in H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$ is homogeneous of degree $\deg a - 2$ or it is zero. We use the induction hypothesis for b.

We proved in [Ma3] (see the proof of Lemma 4.5) that the generators $D_k = D_k(\{Q_i\}, \{P_i\}, \hbar)$, $1 \le k \le l$, of the ideal $\mathcal{I}$ satisfy

$$D_k(\{Q_i\}, \{P_i\}, \hbar).1 = 0. \tag{13}$$

If we let $\hbar$ approach 0 in (13) we obtain the relations

$$D_k(\{q_i\}, \{\sigma_{s_i}\star\}, 0) = 0, \tag{14}$$

$1 \le k \le l$. They generate the whole ideal of relations in the ring $(H^*(G/B) \otimes \mathbb{R}[\{q_i\}], \star)$.

We need to show that if D is an element of $\mathcal{D}$ with the property that

$$D(\{Q_i\}, \{P_i\}, \hbar).1 = 0 \tag{15}$$

then $D \in \mathcal{I}$. Because the map ϕ is degree preserving, we may assume that D is homogeneous and proceed by induction on $\deg D$. If $\deg D = 0$, i.e. D is constant, then (15) implies $D = 0$, hence $D \in \mathcal{I}$. It now follows the induction step. From

$$D.1 = D(\{q_i\}, \{\sigma_{s_i} \star + \hbar q_i \frac{\partial}{\partial q_i}\}, \hbar).1 = 0,$$

for all $\hbar$, we deduce the relation $D(\{q_i\}, \{\sigma_{s_i}\star\}, 0) = 0$ in the ring $(H^*(G/B) \otimes \mathbb{R}[\{q_i\}], \star)$. Consequently we have the following polynomial identity

$$D(\{q_i\}, \{\lambda_i\}, 0) = \sum_k f_k(\{q_i\}, \{\lambda_i\}) D_k(\{q_i\}, \{\lambda_i\}, 0),$$

for certain polynomials f_k. By using the commutation relations (10), we obtain the following identity in $\mathcal{D}$:

$$\begin{aligned} D(\{Q_i\}, \{P_i\}, 0) &\equiv \sum_k f_k(\{Q_i\}, \{P_i\}) D_k(\{Q_i\}, \{P_i\}, 0) \bmod \hbar \\ &\equiv \sum_k f_k(\{Q_i\}, \{P_i\}) D_k(\{Q_i\}, \{P_i\}, \hbar) \bmod \hbar. \end{aligned}$$

In other words,

$$D(\{Q_i\}, \{P_i\}, \hbar) = \sum_k f_k(\{Q_i\}, \{P_i\}) D_k(\{Q_i\}, \{P_i\}, \hbar) + \hbar D'(\{Q_i\}, \{P_i\}, \hbar),$$

for a certain $D' \in \mathcal{D}$, with $\deg D' < \deg D$. From (14) and (15) we deduce that

$$D'(\{Q_i\}, \{P_i\}, \hbar).1 = 0$$

Since $\deg D' < \deg D$, we only have to use the induction hypothesis for D' and get to the desired conclusion.

□

Note that (12) is also an isomorphism of $\mathbb{R}[\{Q_i\}, \hbar]$-modules. Since the actual quantum product $\circ$ satisfies the hypotheses of Theorem 1.2, we deduce that the dimension of $\mathcal{D}/\mathcal{I}$ as an $\mathbb{R}[\{Q_i\}, \hbar]$-module equals $|W|$. Let us consider the "standard monomial basis" $\{[C_w] :$

$w \in W\}$ of $\mathcal{D}/\mathcal{I}$ over $\mathbb{R}[\{Q_i\}, \hbar]$ with respect to a choice of a Gröbner basis of the ideal $\mathcal{I}$ (for more details, see Guest [Gu, section 1] and the references therein). Any C_w is a monomial in $P_1, \ldots, P_l$ and the cosets of the monomials

$$c_w = C_w(\lambda_1, \ldots, \lambda_l), \quad w \in W$$

in $H^*(G/B) = S(\mathfrak{t}^*)/S(\mathfrak{t}^*)^W = \mathbb{R}[\{\lambda_i\}]/I_W$ are a basis. We will need the following result (our proof relies on an idea of Amarzaya and Guest [Am-Gu]):

Proposition 2.2. *There exists a unique basis $\{[\bar{C}_w] : w \in W\}$ of $\mathcal{D}/\mathcal{I}$ over $\mathbb{R}[\{Q_i\}, \hbar]$ with the following properties:*

(i) *for all $w \in W$ the element $\bar{C}_w = \bar{C}_w(\{Q_i\}, \{P_i\}, \hbar)$ of $\mathcal{D}$ is homogeneous of degree $2 \deg c_w$ with respect to the grading defined by (11)*

(ii) *for all $w \in W$ we have*

$$\bar{C}_w(\{0\}, \{\lambda_i\}, \hbar) \equiv c_w \bmod I_W;$$

in particular $\bar{C}_w(\{0\}, \{\lambda_i\}, \hbar) \bmod I_W$ is independent of $\hbar$

(iii) *the elements $(\bar{\Omega}^i_{vw})^{1 \le i \le l}_{v,w \in W}$ of $\mathbb{R}[Q_1, \ldots, Q_l, \hbar]$ determined by*

$$P_i[\bar{C}_w] = \sum_{v \in W} \bar{\Omega}^i_{vw}[\bar{C}_v],$$

are independent of $\hbar$.

Proof. In order to show that such a basis exists, we consider the isomorphism

$$\phi : \mathcal{D}/\mathcal{I} \to H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar]$$

induced by the actual quantum product $\circ$ via Proposition 2.1. The basis $\{[c_w] : w \in W\}$ of the right hand side induces the basis $\{[\bar{C}_w] = \phi^{-1}([c_w]) : w \in W\}$ of $\mathcal{D}/\mathcal{I}$ over $\mathbb{R}[\{Q_i\}, \hbar]$. It is obvious that the latter basis satisfies (i) and (iii). In order to show that it also satisfies (ii), we consider the following commutative diagram:

$$\begin{array}{ccc} \mathcal{D}/\mathcal{I} & \xrightarrow{\phi} & H^*(G/B) \otimes \mathbb{R}[\{q_i\}, \hbar] \\ \psi_1 \searrow & & \swarrow \psi_2 \\ & H^*(G/B) \otimes \mathbb{R}[\hbar] & \end{array}$$

where ψ_2 is the canonical projection and $\psi_1 : \mathcal{D}/\mathcal{I} \to H^*(G/B) \otimes \mathbb{R}[\hbar] = (\mathbb{R}[\{\lambda_i\}]/I_W) \otimes \mathbb{R}[\hbar]$ is given by

$$[D(\{Q_i\}, \{P_i\}, \hbar)] \mapsto [D(\{0\}, \{\lambda_i\}, \hbar)].$$

Note that ψ_1 is well defined, as for any $k = 1, 2, \ldots, l$, the polynomial $D_k(\{0\}, \{\lambda_i\}, \hbar)$ is independent of $\hbar$, being equal to u_k, the k-th fundamental W-invariant polynomial (see [Ma 2, section 3]). We observe that

$$[\bar{C}_w(\{0\}, \{\lambda_i\}, \hbar)] = \psi_1[\bar{C}_w] = \psi_2[c_w] = [c_w],$$

hence condition (ii) is satisfied.

In order to show that there exists at most one such basis, we will use a construction of Amarzaya and Guest [Am-Gu]. Let $\{[\bar{C}_w] : w \in W\}$ be a basis of $\mathcal{D}/\mathcal{I}$ with the properties (i), (ii) and (iii). We can write

$$\bar{C}_w \equiv \sum_{v\in W} U^{vw} C_v \mod \mathcal{I} \tag{16}$$

with $U^{vw} \in \mathbb{R}[\{Q_i\}, \hbar]$. Decompose the matrix $U = (U^{vw})_{v,w\in W}$ as

$$U = U_0 + \hbar U_1 + \ldots + \hbar^k U_k,$$

where $U_0, \ldots, U_k$ have entries in $\mathbb{R}[\{Q_i\}]$. Let us apply ψ_1 to both sides of equation (16) and deduce that in $H^*(G/B) \otimes \mathbb{R}[\hbar] = (\mathbb{R}[\{\lambda_i\}]/I_W) \otimes \mathbb{R}[\hbar]$ we have that

$$[c_w] = \sum_{v\in W} U^{vw}|_{(\text{all } Q_i=0)}[c_v],$$

for all $w \in W$. This implies

$$U^{vw}|_{(\text{all } Q_i=0)} = \delta_{vw},$$

where δ_{vw} is the Kroenecker delta. On the other hand, because any $\bar{C}_w$, C_v, $v, w \in W$, as well as any generator D_i of $\mathcal{I}$ is homogeneous, we deduce that each U^{vw} is homogeneous. We are led to the following property of the matrices U_j:

(a) besides the diagonal of U_0, which is I, the entries of $U_0, U_1, \ldots, U_k$ are homogeneous polynomials with no degree zero term in $Q_1, \ldots, Q_l$

Let us choose an ordering of W which is increasing with respect to $\deg c_w$. In this way, the set $\{[c_w] : w \in W\}$ is a basis of $H^*(G/B)$ consisting of $s_0 = 1$ elements of degree 0, followed by s_1 elements of degree 2, … , followed by s_m elements of degree $2m = \dim G/B$. All matrices involved here appear as block matrices of the type $A = (A_{\alpha\beta})_{1\le\alpha,\beta\le m}$. We will say that a block matrix $A = (A_{\alpha\beta})_{1\le\alpha,\beta\le m}$ is *r-triangular* if $A_{\alpha\beta} = 0$ for all α, β with $\beta - \alpha < r$. From the homogeneity of U^{vw} mentioned above and the fact that $\deg Q_1 = \ldots = \deg Q_l = 4$, we deduce:

(b) the block matrix $U_0 - I$ is 2-triangular
(c) for any $1 \le j \le k$, the block matrix U_j is $(j + 2)$-triangular.

In particular we can assume that $k = m - 2$, hence

$$U = U_0 + \hbar U_1 + \ldots + \hbar^{m-2} U_{m-2}, \tag{17}$$

Consider the matrix $\Omega^i = (\Omega^i_{vw})_{v,w\in W}$ determined by

$$P_i[C_w] = \sum_{v\in W} \Omega^i_{vw}[C_v]. \tag{18}$$

As before, each Ω^i_{vw} is an element of $\mathbb{R}[Q_1, \ldots, Q_l, \hbar]$ which is homogeneous with respect to the grading given by (11). Also, if we apply ψ_1 on both sides of the equation (18), we deduce that in $H^*(G/B) \otimes \mathbb{R}[\hbar] = (\mathbb{R}[\{\lambda_i\}]/I_W) \otimes \mathbb{R}[\hbar]$ we have

$$[\lambda_i][c_w] = \sum_{v} \Omega^i_{vw}|_{(\text{all } Q_i=0)}[c_v].$$

This shows that

$$\Omega^i_{vw}|_{(\text{all } Q_j=0)} \text{ is independent of } \hbar, \text{for all } v,w\in W, 1\leq i\leq l \tag{19}$$

From here on, it will be more convenient to work with the realization of $\mathcal{D}$ given by $Q_i = e^{t_i}$ and $P_i = \hbar\frac{\partial}{\partial t_i}$, $1 \leq i \leq l$. Then Ω^i become matrices whose coefficients are homogeneous polynomials in $e^{t_1},\dots,e^{t_l}$, and $\hbar$. Let us consider the 1-form

$$\Omega = \sum_{i=1}^{l} \Omega_i dt_i. \tag{20}$$

We decompose it as

$$\Omega = \omega + \hbar\theta^{(1)} + \ldots + \hbar^p\theta^{(p)}.$$

From the homogeneity of the entries of Ω_i, as well as from (20) we deduce that:

(d) the block matrix ω is (-1)-triangular
(e) the block matrix $\theta^{(j)}$ is $(j+1)$-triangular, for any $1\leq j\leq p$.

In particular we can assume that $p = m-1$, hence

$$\Omega = \omega + \hbar\theta^{(1)} + \ldots + \hbar^{m-2}\theta^{(m-2)}. \tag{21}$$

Now consider the matrix $\bar{\Omega}^i = (\bar{\Omega}^i_{vw})_{v,w\in W}$ determined by

$$P_i[\bar{C}_w] = \sum_{v\in W}\bar{\Omega}^i_{vw}[\bar{C}_v].$$

Note that if $p\in\mathcal{D}$ is a polynomial $p(e^{t_1},\dots,e^{t_l})$, then we have

$$\hbar\frac{\partial}{\partial t_i}\cdot p = p\cdot\hbar\frac{\partial}{\partial t_i} + \hbar\frac{\partial}{\partial t_i}(p).$$

By using this, we can easily deduce from (16) that

$$\bar{\Omega}^i = U^{-1}\Omega^i U + \hbar U^{-1}\frac{\partial}{\partial t_i}U.$$

Thus the 1-form $\bar{\Omega} = \sum_{i=1}^l \bar{\Omega}_i dt_i$ is given by

$$\bar{\Omega} = U^{-1}\Omega U + \hbar U^{-1}dU.$$

Condition (iii) reads $\bar{\Omega}$ is independent of $\hbar$. From (17) and (21) we can see that this is equivalent to

$$U^{-1}\Omega U + \hbar U^{-1}dU = U_0^{-1}\omega U_0$$

and further to

$$\Omega U + \hbar dU = UU_0^{-1}\omega U_0. \tag{22}$$

We will prove the following claim

Claim. For a given Ω of the type (21) with the properties (d) and (e), the system (22) has at most one solution U of the type (17) with U_j satisfying (a) and (b).

It is obvious that the claim implies that there exists at most one basis $\{[\bar{C}_w] : w\in W\}$ with the properties (i), (ii) and (iii), and the proof is complete.

In order to prove the claim, let us write

$$U = (I + \hbar V_1 + \hbar^2 V_2 + \ldots + \hbar^{m-2} V_{m-2}) V_0$$

where $V_0 = U_0$, $V_1 = U_1 U_0^{-1}, \ldots, V_{m-2} = U_{m-2} U_0^{-1}$. Note that (a), (b) and (c) from above imply:

(f) V_0 is a block matrix whose diagonal is I, such that $V_0 - I$ is 2-triangular, and all entries of V_0 which are not on the diagonal are polynomials with no degree zero term in $e^{t_1}, \ldots, e^{t_l}$,

(g) V_0^{-1} is an upper triangular matrix, its diagonal is I, and all entries of V_0^{-1} which are not on the diagonal are polynomials with no degree zero term in $e^{t_1}, \ldots, e^{t_l}$,

(h) for any $1 \le j \le m-2$, the block matrix V_j is $(j+2)$-triangular and its entries are polynomials with no degree zero term in $e^{t_1}, \ldots, e^{t_l}$.

By identifying the coefficients of powers of $\hbar$, the equation (22) is equivalent to the system consisting of:

$$d(V_0)V_0^{-1} = -\theta^{(1)} + [V_1, \omega] \tag{23}$$

and

$$\begin{aligned} dV_1 &= -\theta^{(2)} + [V_1, \theta^{(1)}] + [V_2, \omega] - V_1[V_1, \omega] \\ dV_i &= -\theta^{(i+1)} - \theta^{(i)} V_1 - \ldots - \theta^{(2)} V_{i+1} + [V_i, \theta^{(1)}] + [V_{i+1}, \omega] - V_i [V_1, \omega] \end{aligned} \tag{24}$$

for $i \ge 2$.

It is convenient to write a block matrix $A = (A_{\alpha\beta})_{1 \le \alpha, \beta \le m}$ as

$$A = A^{[-m]} + \ldots + A^{[-1]} + A^{[0]} + A^{[1]} + \ldots + A^{[m]}$$

where each block matrix $A^{[j]}$ is j-diagonal (i.e. $A^{[j]}_{\alpha\beta} = 0$ whenever $\beta - \alpha \ne j$). Then for any two block matrices A and B we have:

$$(AB)^{[j]} = \sum_k A^{[k]} B^{[j-k]}, \quad [A, B]^{[j]} = \sum_k [A^{[k]}, B^{[j-k]}].$$

By (b), (c), (d) and (e) we can write:

$$\begin{aligned} V_0 &= I + V_0^{[2]} + V_0^{[3]} + \ldots + V_0^{[m]} \\ V_i &= V_i^{[i+2]} + V_i^{[i+3]} + \ldots + V_i^{[m]} \quad (1 \le i \le m-2) \\ \omega &= \omega^{[-1]} + \omega^{[0]} + \omega^{[1]} + \ldots + \omega^{[m]} \\ \theta^{(i)} &= \theta^{(i),[i+1]} + \theta^{(i),[i+2]} + \ldots + \theta^{(i),[m]} \quad (1 \le i \le m-1) \end{aligned}$$

In this way, the system (24) is equivalent to:

$$\begin{aligned} dV_1^{[j]} = -\theta^{(2),[j]} &+ \sum_{3\leq k\leq j-2}[V_1^{[k]},\theta^{(1),[j-k]}] \\ &+ \sum_{4\leq k\leq j+1}[V_2^{[k]},\omega^{[j-k]}] \\ &- \sum_{2\leq k\leq j-3}\sum_{3\leq l\leq k+1} V_1^{[j-k]}[V_1^{[l]},\omega^{[k-l]}] \end{aligned}$$

$$\begin{aligned} dV_i^{[j]} = -\theta^{(i+1),[j]} &- \sum_{3\leq k\leq j-i-1}\theta^{(i),[j-k]}V_1^{[k]} \\ &- \sum_{i+3\leq k\leq j-3}\theta^{(2),[j-k]}V_{i+1}^{[k]} \\ &+ \sum_{i+2\leq k\leq j-2}[V_i^{[k]},\theta^{(1),[j-k]}] \\ &+ \sum_{i+3\leq k\leq j+1}[V_{i+1}^{[k]},\omega^{[j-k]}] \\ &- \sum_{2\leq k\leq j-i-2}\sum_{3\leq l\leq k+1} V_i^{[j-k]}[V_1^{[l]},\omega^{[k-l]}], \end{aligned}$$

where $i\geq 2$. Define the total order on the matrices $V_i^{[j]}$, $j\geq i+2$ as follows: $V_{i_1}^{[j_1]} < V_{i_2}^{[j_2]}$ if and only if $j_1-i_1 < j_2-i_2$ or $j_1-i_1 = j_2-i_2$ and $j_1<j_2$. We note that the system from above is of the form:

$$dV_i^{[j]} = \text{ expression involving } V_{i'}^{[j']} > V_i^{[j]},$$

for $i\geq 1$ and $j\geq i+2$. Because all coefficients of the matrices $V_i^{[j]}$ are polynomials with no degree zero term in $e^{t_1},\dots,e^{t_l}$, we deduce inductively — starting with $V_{m-2}^{[m]}$ — that there exists at most one solution $V_i^{[j]}$, $i\geq 1$, $j\geq i+2$, of the system. It remains to show that there exists at most one V_0 which satisfies both the condition (f) and the equation (23). If V_0' is another solution, then a simple calculation shows that

$$d(V_0^{-1}V_0') = 0.$$

By condition (f), the matrix $V_0^{-1}V_0'$ has the diagonal I and any entry of it which is not on the diagonal is a polynomial with no degree zero term in $e^{t_1},\dots,e^{t_l}$. So $V_0^{-1}V_0' = I$, which means $V_0 = V_0'$.

□

Now we can prove our main result:

Proof of Theorem 1.2 Let $\star$ be a product with the properties stated in Theorem 1.2. Consider the isomorphism of $\mathcal{D}$-modules

$$\phi : \mathcal{D}/\mathcal{I} \to H^*(G/B)\otimes\mathbb{R}[\{q_i\},\hbar]$$

given by Proposition 2.1. The basis $\{[c_w] : w \in W\}$ of the right hand side induces the basis $\{[\bar{C}_w] = \phi^{-1}([c_w]) : w \in W\}$ of $\mathcal{D}/\mathcal{I}$ over $\mathbb{R}[\{Q_i\}, h]$. It is obvious that the latter satisfies the hypotheses (i) and (iii) of Proposition 2.2. We show that it also satisfies (ii) by using the argument already employed in the first part of the proof of Proposition 2.2. Now from Proposition 2.2, we deduce that

$$[\bar{C}_w] = [\hat{C}_w],$$

for $w \in W$, where the basis $\{[\hat{C}_w] : w \in W\}$ is induced by the actual quantum product $\circ$. Now, since ϕ is an isomorphism of $\mathcal{D}$-modules, $\phi([\bar{C}_w]) = [c_w]$ and $\phi(P_i) = [\lambda_i]$, we deduce that the matrix of $[\lambda_i]\star$ with respect to the basis $\{[c_w] : w \in W\}$ is the same as the matrix of P_i with respect to the basis $\{[\bar{C}_w] : w \in W\}$. Consequently we have

$$[\lambda_i] \star a = [\lambda_i] \circ a,$$

for all $a \in H^*(G/B) \otimes \mathbb{R}[q_1, \dots, q_l]$. Hence the products $\star$ and $\circ$ are the same. $\square$

3. Quantization map for Fl_n

In the case $G = SL(n, \mathbb{C})$, the resulting flag manifold is Fl_n, which is the space of all complete flags in $\mathbb{C}^n$. Borel's presentation (see eq. (1)) in this case reads

$$H^*(Fl_n) = \mathbb{R}[\lambda_1, \dots, \lambda_{n-1}]/(I_n)_{\geq 2},$$

where $(I_n)_{\geq 2}$ denotes the ideal generated by the nonconstant symmetric polynomials of degree at least 2 in the variables

$$x_1 := -\lambda_1, x_2 := \lambda_1 - \lambda_2, \dots, x_{n-1} := \lambda_{n-2} - \lambda_{n-1}, x_n := \lambda_{n-1}.$$

Equivalently, we have

$$H^*(Fl_n) = \mathbb{R}[x_1, \dots, x_n]/I_n$$

where I_n denotes the ideal generated by the nonconstant symmetric polynomials of degree at least 1 in the variables $x_1, \dots, x_n$. For any $k \in \{0, 1, \dots, n\}$ we consider the polynomials $e_0^k, \dots, e_k^k$ in the variables $x_1, \dots, x_k$, which can be described by

$$\det \left[\begin{pmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & x_k \end{pmatrix} + \mu I_k \right] = \sum_{i=0}^{n} e_i^k \mu^{k-i}.$$

For $i_1, \dots, i_{n-1} \in \mathbb{Z}$ such that $0 \leq i_j \leq j$, we define

$$e_{i_1 \dots i_{n-1}} = e_{i_1}^1 \dots e_{i_{n-1}}^{n-1}.$$

These are called the *standard elementary monomials.* It is known (see e.g. [Fo-Ge-Po, Proposition 3.4]) that the set $\{[e_{i_1 \dots i_{n-1}}] \ : \ 0 \leq i_j \leq j\}$ is a basis of $H^*(Fl_n)$.

We also consider the polynomials[4] $\hat{e}_0^k, \dots, \hat{e}_k^k$ in the variables $x_1, \dots, x_k, q_1, \dots, q_{k-1}$, which are described by

$$\det \left[\begin{pmatrix} x_1 & q_1 & 0 & \dots & 0 \\ -1 & x_2 & q_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & x_{k-1} & q_{k-1} \\ 0 & \dots & 0 & -1 & x_k \end{pmatrix} + \mu I_k \right] = \sum_{i=0}^{k} \hat{e}_i^k \mu^{k-i}.$$

For $i_1, \dots, i_{n-1}$ such that $0 \le i_j \le j$, we define the *quantum standard elementary monomials*

$$\hat{e}_{i_1 \dots i_{n-1}} = \hat{e}_{i_1}^1 \dots, \hat{e}_{i_{n-1}}^{n-1}.$$

By a theorem of Ciocan-Fontanine [Ci] (in fact Kim's theorem for $G = SL(n, \mathbb{C})$, see section 1), we have the following isomorphism of $\mathbb{R}[q_1, \dots, q_{n-1}]$-algebras

$$(H^*(Fl_n) \otimes \mathbb{R}[q_1, \dots, q_{n-1}], \circ) \simeq QH^*(Fl_n) := \mathbb{R}[x_1, \dots x_n, q_1, \dots, q_{n-1}]/\langle \hat{e}_1^n, \dots, \hat{e}_n^n \rangle, \tag{25}$$

which is canonical, in the sense that $[x_i]$ is mapped to $[x_i]_q$. According to [Fo-Ge-Po], we will call this the *quantization map*. Since the conditions (6) and (7) are satisfied, we deduce that $\{[\hat{e}_{i_1 \dots i_{n-1}}]_q \ : \ 0 \le i_j \le j\}$ is a basis of $QH^*(Fl_n)$ over $\mathbb{R}[q_1, \dots, q_{n-1}]$. We also point out the obvious fact that $\{[e_{i_1 \dots i_{n-1}}] \ : \ 0 \le i_j \le j\}$ is a basis of $H^*(Fl_n) \otimes \mathbb{R}[q_1, \dots, q_{n-1}]$ over $\mathbb{R}[q_1, \dots, q_{n-1}]$. The goal of this section is to give a different proof to the following theorem of Fomin, Gelfand, and Postnikov.

Theorem 3.1. (see [Fo-Ge-Po, Theorem 1.1]). *The quantization map described by equation (25) sends $[e_{i_1 \dots i_{n-1}}]$ to $[\hat{e}_{i_1 \dots i_{n-1}}]_q$.*

The main instrument of our proof is the $\mathcal{D}$-module $\mathcal{D}/\mathcal{I}$ defined in section 2. In this case (i.e. $G = SL(n, \mathbb{C})$) we can describe it explicitly, as follows: $\mathcal{D}$ is the (noncommutative) Heisenberg algebra defined at the beginning of section 2, where $l = n - 1$. The left ideal $\mathcal{I}$ of $\mathcal{D}$ is generated by $\mathcal{E}_1^n, \dots, \mathcal{E}_{n-1}^n$, where

$$\det \left[\begin{pmatrix} -P_1 & Q_1 & 0 & \dots & 0 \\ -1 & P_1 - P_2 & Q_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & P_{n-2} - P_{n-1} & Q_{n-1} \\ 0 & \dots & 0 & -1 & P_{n-1} \end{pmatrix} + \mu I_n \right] = \sum_{i=0}^{n} \mathcal{E}_i^n \mu^{n-i}.$$

In fact we will need more general elements of $\mathcal{D}$, namely, for each $k \in \{1, \dots, n-1\}$, we consider the elements $\mathcal{E}_i^k$ of $\mathcal{D}$, with $0 \le i \le k$, given by

$$\det \left[\begin{pmatrix} -P_1 & Q_1 & 0 & \dots & 0 \\ -1 & P_1 - P_2 & Q_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & P_{k-2} - P_{k-1} & Q_{k-1} \\ 0 & \dots & 0 & -1 & P_{k-1} - P_k \end{pmatrix} + \mu I_k \right] = \sum_{i=0}^{k} \mathcal{E}_i^k \mu^{k-i}.$$

[4]These are the polynomials E_i^k of [Fo-Ge-Po].

One can easily see that when we expand the determinant in the left hand side of the last equation, we will have no occurrence of P_jQ_j or Q_jP_j, $1 \le j \le k-1$. This means that the lack of commutativity of Q_j and P_j creates no ambiguity in the definition of $\mathcal{E}_1^n, \dots, \mathcal{E}_{n-1}^n$. We can also deduce that each of $\mathcal{E}_1^k, \dots, \mathcal{E}_k^k$ is a linear combination of monomials in the variables $\{P_1, \dots, P_k, Q_1, \dots, Q_{k-1}\}$, with no ocurrence of P_jQ_j or Q_jP_j (i.e. the order of factors in each monomial is not important). As a consequence, the following recurrence formula [Fo-Ge-Po, equation (3.5)] still holds:

$$\mathcal{E}_i^k = \mathcal{E}_i^{k-1} + X_k\mathcal{E}_{i-1}^{k-1} + Q_{k-1}\mathcal{E}_{i-2}^{k-2}, \tag{26}$$

where X_k stands for $P_{k-1} - P_k$ and, by convention, $\mathcal{E}_j^k = 0$, unless $0 \le j \le k$. It is worth mentioning the following commutation relations, which will be used later:

$$[X_k, \mathcal{E}_j^l] = 0, \quad [Q_k, \mathcal{E}_j^l] = 0, \tag{27}$$

whenever $l \le k-1$. We also note that $\mathcal{E}_0^k = 1$ and $\mathcal{E}_1^k = -P_k$ (where P_n is by convention equal to 0). We will prove the following result.

Lemma 3.2. *The elements $\mathcal{E}_1^k, \dots, \mathcal{E}_{k-1}^k$ of $\mathcal{D}$ commute with each other.*

Proof. Consider the coordinates $s_0, \dots, s_{k-1}$ on $\mathbb{R}^k$. Following [Kim-Joe], we consider the differential operators $D_j(\hbar\frac{\partial}{\partial s_0}, , \dots, \hbar\frac{\partial}{\partial s_{k-1}}, e^{s_1-s_0}, \dots, e^{s_{k-1}-s_{k-2}})$ given by

$$\det\left[\begin{pmatrix} \hbar\frac{\partial}{\partial s_0} & e^{s_1-s_0} & 0 & \dots & 0 \\ -1 & \hbar\frac{\partial}{\partial s_1} & e^{s_2-s_1} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & -1 & \hbar\frac{\partial}{\partial s_{k-2}} & e^{s_{k-1}-s_{k-2}} \\ 0 & \dots & 0 & -1 & \hbar\frac{\partial}{\partial s_{k-1}} \end{pmatrix} + \mu I_k\right] = \sum_{i=0}^{k} D_i^k \mu^{k-i}.$$

By [Kim-Joe, Proposition 1], we have $[D_i^k, D_j^k] = 0$ for all $0 \le i, j \le k$. In order to prove our lemma, it is sufficient to note that if we make the change of coordinates

$$s_1 - s_0 = t_1, \dots, s_{k-1} - s_{k-2} = t_{k-1}, -s_{k-1} = t_k,$$

we obtain

$$\hbar\frac{\partial}{\partial s_0} = -\hbar\frac{\partial}{\partial t_1} = -P_1, \hbar\frac{\partial}{\partial s_1} = \hbar\frac{\partial}{\partial t_1} - \hbar\frac{\partial}{\partial t_2} = P_1 - P_2, \dots, \hbar\frac{\partial}{\partial s_{k-1}} = \hbar\frac{\partial}{\partial t_{k-1}} - \hbar\frac{\partial}{\partial t_k} = P_{k-1} - P_k,$$

where we have used the presentation of $\mathcal{D}$ given by $P_i = \hbar\frac{\partial}{\partial t_i}, Q_i = e^{t_i}$, $1 \le i \le n-1$. $\square$

The following technical result will be needed later.

Lemma 3.3. *We have*

$$[\mathcal{E}_{j+1}^{k+1}, \mathcal{E}_i^k] = [\mathcal{E}_{i+1}^{k+1}, \mathcal{E}_j^k]. \tag{28}$$

Proof. We prove this by induction on $k \ge 0$. For $k = 0$, the equation is obvious (by the convention made above, we have $\mathcal{E}_0^j = 0$). It follows the induction step. We use the recurrence formula (26). This gives

$$[\mathcal{E}_{j+1}^{k+1}, \mathcal{E}_i^k] = [\mathcal{E}_{j+1}^k + X_{k+1}\mathcal{E}_j^k + Q_k\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^k] = [Q_k\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^k].$$

We continue by using again equation (26) and obtain

$$\begin{aligned}
&[Q_k\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^{k-1} + X_k\mathcal{E}_{i-1}^{k-1} + Q_{k-1}\mathcal{E}_{i-2}^{k-2}] \\
&= [Q_k, X_k]\mathcal{E}_{i-1}^{k-1}\mathcal{E}_{j-1}^{k-1} + [Q_k\mathcal{E}_{j-1}^{k-1}, Q_{k-1}\mathcal{E}_{i-2}^{k-2}] \\
&= [Q_k, X_k]\mathcal{E}_{i-1}^{k-1}\mathcal{E}_{j-1}^{k-1} + Q_k[\mathcal{E}_{j-1}^{k-1}, Q_{k-1}\mathcal{E}_{i-2}^{k-2}] \\
&= [Q_k, X_k]\mathcal{E}_{i-1}^{k-1}\mathcal{E}_{j-1}^{k-1} + Q_k[\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^{k} - \mathcal{E}_i^{k-1} - X_k\mathcal{E}_{i-1}^{k-1}] \\
&= [Q_k, X_k]\mathcal{E}_{i-1}^{k-1}\mathcal{E}_{j-1}^{k-1} + Q_k([\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^{k}] - [\mathcal{E}_{j-1}^{k-1}, X_k\mathcal{E}_{i-1}^{k-1}]) \\
&= [Q_k, X_k]\mathcal{E}_{i-1}^{k-1}\mathcal{E}_{j-1}^{k-1} + Q_k[\mathcal{E}_{j-1}^{k-1}, \mathcal{E}_i^{k}]
\end{aligned}$$

Here we have used the commutation relations (27) several times. Similarly, we obtain

$$[\mathcal{E}_{i+1}^{k+1}, \mathcal{E}_j^k] = [Q_k, X_k]\mathcal{E}_{j-1}^{k-1}\mathcal{E}_{i-1}^{k-1} + Q_k[\mathcal{E}_{i-1}^{k-1}, \mathcal{E}_j^k].$$

We use the induction hypothesis to finish the proof. □

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let ω_k denote the matrix of multiplication by $[y_k]_q$ with respect to the basis $\{[\hat{e}_{i_1\dots i_{n-1}}]_q \ : \ 0 \le i_j \le j\}$ of $QH^*(Fl_n)$ (see equation (25)). More precisely, the entries of ω_i are polynomials in $q_1, \dots, q_{n-1}$, determined by

$$[y_k]_q[\hat{e}_{i_1\dots i_{n-1}}]_q = \sum_{l_1,\dots,l_{n-1}} \omega_k^{i_1\dots i_{n-1}, l_1\dots l_{n-1}}[\hat{e}_{l_1\dots l_{n-1}}]_q. \tag{29}$$

According to Corollary 1.3, it is sufficient to show that

$$\frac{\partial}{\partial t_i}\omega_j = \frac{\partial}{\partial t_j}\omega_i, \tag{30}$$

for $1 \le i, j \le n-1$, where as usually, we use the convention $q_i = e^{t_i}$. For $i_1, \dots, i_{n-1}$ such that $0 \le i_j \le j$, we consider

$$\mathcal{E}_{i_1\dots i_{n-1}} := \mathcal{E}_{i_1}^1\mathcal{E}_{i_2}^2 \dots \mathcal{E}_{i_{n-1}}^{n-1}.$$

In order to prove equation (30), it is sufficient to prove the following claim.

Claim. In $\mathcal{D}/\mathcal{I}$ we have

$$[P_k][\mathcal{E}_{i_1\dots i_{n-1}}] = \sum_{l_1,\dots,l_{n-1}} \Omega_k^{i_1\dots i_{n-1}, l_1\dots l_{n-1}}[\mathcal{E}_{l_1\dots l_{n-1}}], \tag{31}$$

where each $\Omega_k^{i_1\dots i_{n-1}, l_1\dots l_{n-1}}$ is obtained from $\omega_k^{i_1\dots i_{n-1}, l_1\dots l_{n-1}}$ by the modification $Q_i \mapsto q_i$.

Indeed, if we make the usual identifications $P_k = \hbar\frac{\partial}{\partial t_k}$, $Q_k = e^{t_k}$, $1 \le k \le n-1$, then (31) implies that the connection

$$d + \sum_{k=1}^{n-1} \frac{1}{\hbar}\Omega_k dt_k$$

is flat (see e.g. [Gu, Proposition 1.1]) for all values of $\hbar$, which implies (30). The proof of the claim relies on a noncommutative version of the quantum straightening algorithm of Fomin,

Gelfand, and Postnikov [Fo-Ge-Po]. The key equation is the following.

$$\mathcal{E}_i^k\mathcal{E}_{j+1}^{k+1} + \mathcal{E}_{i+1}^k\mathcal{E}_j^k + Q_k\mathcal{E}_{i-1}^{k-1}\mathcal{E}_j^k = \mathcal{E}_j^k\mathcal{E}_{i+1}^{k+1} + \mathcal{E}_{j+1}^k\mathcal{E}_i^k + Q_k\mathcal{E}_{j-1}^{k-1}\mathcal{E}_i^k. \tag{32}$$

We note that this is the same as equation (3.6) in [Fo-Ge-Po]. The difference is that here we work in the algebra $\mathcal{D}$, which is not commutative, so it is not *a priori* clear that (32) still holds. In order to prove it, we use equation (26) twice and obtain:

$$(\mathcal{E}_{j+1}^{k+1} - \mathcal{E}_{j+1}^k)\mathcal{E}_i^k = (X_{k+1}\mathcal{E}_j^k + Q_k\mathcal{E}_{j-1}^{k-1})\mathcal{E}_i^k,$$

and

$$(\mathcal{E}_{i+1}^{k+1} - \mathcal{E}_{i+1}^k)\mathcal{E}_j^k = (X_{k+1}\mathcal{E}_i^k + Q_k\mathcal{E}_{i-1}^{k-1})\mathcal{E}_j^k.$$

If we subtract the second equation from the first one, we obtain:

$$\mathcal{E}_{i+1}^{k+1}\mathcal{E}_j^k - \mathcal{E}_{j+1}^{k+1}\mathcal{E}_i^k = \mathcal{E}_{i+1}^k\mathcal{E}_j^k - \mathcal{E}_{j+1}^k\mathcal{E}_i^k + Q_k(\mathcal{E}_{i-1}^{k-1}\mathcal{E}_j^k - \mathcal{E}_{j-1}^{k-1}\mathcal{E}_i^k).$$

Now the left hand side can be written as

$$\mathcal{E}_j^k\mathcal{E}_{i+1}^{k+1} - \mathcal{E}_i^k\mathcal{E}_{j+1}^{k+1} + [\mathcal{E}_{i+1}^{k+1}, \mathcal{E}_j^k] - [\mathcal{E}_{j+1}^{k+1}, \mathcal{E}_i^k] = \mathcal{E}_j^k\mathcal{E}_{i+1}^{k+1} - \mathcal{E}_i^k\mathcal{E}_{j+1}^{k+1},$$

where we have used Lemma 3.3. Equation (32) has been proved. Now we can use it exactly like in the commutative situation, described in [Fo-Ge-Po], in order to obtain the expansion of the product of $P_k = -\mathcal{E}_1^k$ and $\mathcal{E}_{i_1\dots i_{n-1}} = \mathcal{E}_{i_1}^1\dots\mathcal{E}_{i_{n-1}}^{n-1}$. More precisely, we begin with

$$P_k\mathcal{E}_{i_1\dots i_{n-1}} = \mathcal{E}_{i_1}^1\dots\mathcal{E}_{i_{k-1}}^{k-1}P_k\mathcal{E}_{i_k}^k\mathcal{E}_{i_{k+1}}^{k+1}\dots\mathcal{E}_{i_{n-1}}^{n-1} = -\mathcal{E}_{i_1}^1\dots\mathcal{E}_{i_{k-1}}^{k-1}\mathcal{E}_1^k\mathcal{E}_{i_k}^k\mathcal{E}_{i_{k+1}}^{k+1}\dots\mathcal{E}_{i_{n-1}}^{n-1},$$

and then we use (32) repeatedly. The resulting coefficients in the final expansion will be the same as in the commutative situation. This finishes the proof of the claim, and also of Theorem 3.1. □

References

[Am-Gu] A. Amarzaya and M. A. Guest, *Gromov-Witten invariants of flag manifolds, via D-modules*, Jour. London Math. Soc. (2) **72** (2005), 121–136

[Be-Ge-Ge] I. N. Bernstein, I. M. Gelfand and S. I. Gelfand, *Schubert cells and cohomology of the space G/P*, Russian Math. Surveys **28** (1973), 1–26

[Bo] A. Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes des groupes de Lie compacts*, Ann. of Math. **57** No. 2 (1953), 115–207

[Ci] I. Ciocan-Fontanine, *The quantum cohomology ring of flag varieties*, Trans. Amer. Math. Soc. **351** (1999), no. 7, 2695–2729

[Du] B. Dubrovin, *The geometry of 2D topological field theories*, Integrable Systems and Quantum Groups, Lecture Notes in Mathematics, Vol. 1620, Springer-Verlag, New York, 1996, 120–348

[Fo-Ge-Po] S. Fomin, S. Gelfand, and A. Postnikov, *Quantum Schubert polynomials*, J. Amer. Math. Soc., **10** (1997), 565–596

[Fu-Pa] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology* Algebraic geometry—Santa Cruz 1995, Proc. Sympos. Pure Math., 62, Part 2, editors J. Kollar, R. Lazarsfeld and D.R. Morrison, 1997, 45–96

[Fu-Wo] W. Fulton and C. Woodward, *On the quantum product of Schubert classes*, J. Algebraic Geom. **13** (2004), 641–661

[Gu] M. A. Guest, *Quantum cohomology via D-modules*, Topology **44** (2005), 263–281

[Ir] H. Iritani, *Quantum D-module and equivariant Floer theory for free loop spaces*, preprint `math.DG/0410487`

[Kim] B. Kim, *Quantum cohomology of flag manifolds G/B and quantum Toda lattices*, Ann. of Math. **149** (1999), 129–148

[Kim-Joe] B. Kim and D. Joe, *Equivariant mirrors and the Virasoro conjecture for flag manifolds*, Int. Math. Res. Not. **15** (2003), 859–882

[Ma1] A.-L. Mare, *On the theorem of Kim concerning $QH^*(G/B)$*, Integrable systems, topology and physics, editors M. Guest, R. Miyaoka and Y. Ohnita, Contemp. Math. 309, Amer. Math. Soc. (2002), 151-163

[Ma2] A.-L. Mare, *Polynomial representatives of Schubert classes in $QH^*(G/B)$*, Math. Res. Lett. **9** (2002), 757–770

[Ma3] A.-L. Mare, *Relations in the quantum cohomology ring of G/B*, Math. Res. Lett. **11** (2004), 35–48

[Ma4] A.-L. Mare, *The combinatorial quantum cohomology ring of G/B*, Jour. Alg. Comb. **21** (2005), 331–349

[Pe] D. Peterson, *Lectures on quantum cohomology of G/P*, M.I.T. 1996

[Si-Ti] B. Siebert and G. Tian, *On quantum cohomology rings of Fano manifolds and a formula of Vafa and Intriligator*, Asian J. Math. **1** (1997), 679-695

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